

New Solutions of the Inflationary Flow Equations

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ABSTRACT: The inflationary flow equations are a frequently used method of surveying the space of inflationary models. In these applications the infinite hierarchy of differential equations is truncated in a way which has been shown to be equivalent to restricting the set of models considered to those characterized by polynomial inflaton potentials. This paper explores a different method of solving the flow equations, which does not truncate the hierarchy and in consequence covers a much wider class of models while retaining the practical usability of the standard approach.

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1. Introduction

The inflationary flow equations, introduced by Kinney [1] following earlier work by Hoffman and Turner [2] are frequently used as a means of surveying the space of scalar field theories describing inflation. The flow equations form an infinite hierarchy of ordinary differential equations. They provide a convenient framework for parameterizing the space of inflationary solutions of Einstein's equations coupled to a single inflaton with canonical kinetic terms. These equations form the basis of Monte-Carlo reconstruction of the inflaton potential [3], as well as other studies which explore the space of inflationary models [6]-[14].

Practical applications of the flow equations involve truncating the infinite hierarchy so as to obtain a closed set of equations which can be solved. The procedure introduced by Kinney [1] and used by all subsequent studies¹ defines subspaces of

¹Some interesting alternative approaches to characterizing the space of inflationary models can be found in the paper of Ramirez and Liddle [15].

solutions characterized by all Hubble flow parameters vanishing apart from a finite number. It was shown by Liddle [16] that this procedure corresponds to restricting the set of all inflaton potentials to polynomials of order related to the number of flow parameters allowed to assume non-zero values.

The flow equations themselves do not make any assumptions about the potential energy density which defines a specific model. Each solution of the flow equations however corresponds to some definite potential, which can easily be obtained. Thus subspaces of the space of all solutions correspond to definite classes of potentials; dynamical information enters the flow equations algorithm by the means chosen for truncating the hierarchy. That is the point when the class of potentials to be scanned is determined.

While straightforward (and adequate for many purposes), the truncation scheme introduced by Kinney [1] and universally employed in subsequent studies of the flow equations excludes some interesting models of inflation, for example those involving exponentials of the inflaton field. Such cases arise in some supergravity or string motivated models [17, 18], and so it would be nice to be able to broaden the scheme so that they could be included. One may argue that for studying physical effects which are sensitive only to a limited range of inflaton values a polynomial approximation for the potential may be all that is needed, but at least from the theoretical point of view one would like to understand the choices involved. Furthermore, in some cases discussed in the literature, such as models involving potentials with sharp “features” [19]-[22], the polynomial approximation is by definition unlikely to be sufficient.

In view of the above it becomes interesting to consider alternative schemes of dealing with the flow equations. The purpose of this note is to offer a method of solving the hierarchy, which does not set an infinite number of flow parameters to zero. Indeed, all the flow parameters are non-zero in this approach, and in consequence this method does not restrict the space of generated potentials to polynomials in the inflaton field. The hierarchy effectively terminates because flow parameters of higher order are expressed algebraically in terms of a finite number lower order ones as a consequence of a condition which requires that a flow parameter of some order be constant.

The immediate question is then how the new set of potentials is related to the set of polynomial potentials scanned in Kinney’s procedure. It is obvious from the new termination condition (introduced in section 3) that the space of potentials accessible with the new method is strictly a superset of the old one. One way to see the impact of the new scheme would be to repeat Kinney’s numerical study and see how the set of generated models broadens the regions appearing in [1] (and in following studies of this type). This is clearly interesting to explore. This note however focuses on some simple insights which can be gained by analytically solving the new termination conditions at low levels, very much in the spirit of Liddle’s analytic solution of the original scheme [16]. Unlike that case however, a complete analytic solution is possible only in the two lowest orders: at higher orders one needs to resort to numerical methods. The analytic solutions described in section 4 show that the new scheme, in accordance with expectations, brings in non-polynomial potentials. In particular, the lowest order solution describes power law inflation, which is outside the standard scheme since it requires an exponential potential. The next order leads to some interesting cases which have appeared in the literature in various contexts [23]–[25]. They include so-called “ultra-slow-roll inflation” [26], which has the second Hubble parameter $\eta_H = 3$. At higher orders one can write down some special solutions analytically, but for practical applications numerical integration is required. One can proceed to integrate the flow equations directly, or alternatively integrate the termination condition, which is a single ordinary differential equation of order $M + 1$, where M is the order of the flow parameter required to be constant. The second approach directly gives $H(\phi)$, the Hubble parameter as a function of the inflaton. Both options can be implemented in a straightforward manner and from a technical point of view neither requires anything beyond what is used in investigations using Kinney’s truncation [1].

The inflationary flow equations, as well as their standard truncation are described in section 2. The new scheme is presented in section 3 and some analytic solutions of the termination conditions are described in section 4. Some closing comments are offered in section 5.

2. Truncating the Flow Equations

The inflationary flow equations introduced in [1] assume that inflation is driven by a single scalar field described by an effective action of the form

$$S = - \int d^4x \sqrt{-g} \left(\frac{1}{2} (\partial\phi)^2 + V(\phi) \right) . \quad (2.1)$$

For spatially homogeneous field configurations Einstein equations reduce to

$$\dot{\rho} = -3H(p + \rho) \quad (2.2)$$

$$3M_P^2 H^2 = \rho , \quad (2.3)$$

where

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (2.4)$$

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) . \quad (2.5)$$

Here M_P is the reduced Planck mass ($M_P^2 = 1/8\pi G$), the dot indicates a time derivative and $H \equiv \dot{a}/a$.

It is convenient to write these equations in first order form, treating ϕ as the evolution parameter in place of t . From (2.2) – (2.5) it follows that

$$\dot{\phi} = -2M_P^2 H'(\phi) , \quad (2.6)$$

where the prime denotes a derivative with respect to ϕ . Using this and (2.5) in (2.3) gives

$$2M_P^4 H'(\phi)^2 = 3M_P^2 H^2(\phi) - V(\phi) . \quad (2.7)$$

This is the Hamilton-Jacobi form of the field equations [27]-[30].

The fundamental indicator of inflation is the first Hubble flow parameter

$$\epsilon_H = 2M_P^2 \left(\frac{H'}{H} \right)^2 , \quad (2.8)$$

where the prime indicates a derivative with respect to the inflaton field. The basic property of ϵ_H is that

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon_H) , \quad (2.9)$$

which shows that the Universe is inflating if and only if $\epsilon_H < 1$.

The number of e-folds at some time t before the end of inflation at time t_f is given by

$$N = \int_t^{t_f} H dt , \quad (2.10)$$

so one has $dN = -H dt$. This convention defines N as the number of e-folds before the end of inflation at $N = 0$. Thus as time flows forward, N decreases. From (2.10) it follows that

$$\frac{d}{dN} = -\frac{\dot{\phi}}{H} \frac{d}{d\phi} , \quad (2.11)$$

which can be rewritten using (2.6) as

$$\frac{d}{dN} = 2M_P^2 \frac{H'}{H} \frac{d}{d\phi} . \quad (2.12)$$

By direct computation one then finds

$$\frac{d\epsilon_H}{dN} = -2\epsilon_H(\epsilon_H - \eta_H) , \quad (2.13)$$

where

$$\eta_H = 2M_P^2 \frac{H''}{H} \quad (2.14)$$

is the second flow parameter.

The derivative of η_H involves the third derivative of H , which motivates the introduction of another dimensionless flow parameter. Proceeding in this way all higher derivatives of the Hubble parameter appear and an infinite hierarchy of differential equations is generated. It can be described compactly by introducing the infinite sequence of Hubble flow parameters [1] defined as

$$\begin{aligned} \lambda_0 &= 2M_P^2 \left(\frac{H'}{H} \right)^2 \\ \lambda_k &= (2M_P^2)^k \frac{(H')^{k-1}}{H^k} \frac{d^{k+1}H}{d\phi^{k+1}} , \quad k \geq 1 , \end{aligned} \quad (2.15)$$

so that $\lambda_0 = \epsilon_H$ and $\lambda_1 = \eta_H$. The flow equations can now be written as

$$\frac{d\lambda_0}{dN} = 2\lambda_0(\lambda_0 - \lambda_1) \quad (2.16)$$

$$\frac{d\lambda_k}{dN} = \left(-k\lambda_0 + (k-1)\lambda_1 \right) \lambda_k + \lambda_{k+1} , \quad k \geq 1 . \quad (2.17)$$

This is an infinite hierarchy of differential equations for the Hubble flow parameters λ_k . Solutions of these equations for which $\lambda_0 < 1$ for a sufficiently long time describe inflating spacetimes of interest in cosmology.

It was emphasized by Liddle [16] that the flow equations do not reflect any specific choice of potential, since their derivation does not make use of the Hamilton-Jacobi equation. This is consistent with their purpose, which is to describe inflationary solutions for canonical scalar field theories without prejudice. While the equations themselves do not involve a choice of scalar potential, any *specific* solution of (2.17) corresponds to a specific scalar field theory. This is because once ϵ_H is found as a function of N by solving the flow equations one can calculate $H(\phi)$ (up to an overall scale)². This in turn determines the scalar potential via the Hamilton-Jacobi equation (2.7):

$$V(\phi) = 3M_P^2 H(\phi)^2 - 2M_P^4 H'(\phi)^2 . \quad (2.18)$$

Thus every solution of the flow equations determines the corresponding scalar potential (up to an overall energy scale).

The set of all solutions of the flow equations is identical to the set of all solutions of the Hamilton-Jacobi equations for all choices of $V(\phi)$. In that sense the full space of solutions does not reflect any choice of dynamics – it conveniently parameterizes the outcome of all the possible choices. Choosing a class of solutions (a subspace of all solutions) is however tantamount to a statement of dynamics, and this is what practical applications of the flow equations do.

The procedure introduced by Kinney [1], and elaborated on by many authors, involves truncating the infinite hierarchy by setting, for some integer M ,

$$\lambda_k = 0 , \quad k \geq M . \quad (2.19)$$

This yields a closed set of differential equations for $\lambda_0 \dots \lambda_M$. It is important to note that truncating the flow equations is not an approximation: solutions to the truncated set of equations are exact, but they span a subset of all the solutions to the flow equations. Thus truncation is equivalent to restricting the set of all possible

²This also requires relating N and ϕ , which can be done using $dN = 2M_P^2 \frac{H}{H'} d\phi$, which follows from (2.10) and (2.6).

potentials to some subset. This fact was made explicit by Liddle [16] who observed that the truncation condition (2.19) could (using (2.15)) be written as

$$\frac{d^{M+1}H}{d\phi^{M+1}} = 0 \ , \quad (2.20)$$

which makes it plain that solutions of the flow equations are polynomials of order M :

$$H(\phi) = \sum_{k=0}^M a_k \phi^k \ . \quad (2.21)$$

Using this in (2.18) implies that the corresponding scalar potentials are polynomials in ϕ of order $2M$. One can survey a large space of potentials by truncating the flow equations at a high level, i.e. by taking M large in (2.19).

3. New Solutions of the Flow Equations

While the class of polynomial potentials appearing in the standard treatments of the flow equations may be sufficient for most practical purposes, from a theoretical perspective it seems somewhat limited. Indeed, from the point of view of embedding inflationary scalar field theories in string theory it seems that this restriction is quite severe, since non-polynomial contributions to scalar potentials are quite common in that setting [17, 18]. It turns out however that a very simple modification of the standard truncation of the flow hierarchy significantly broadens the set of potentials covered without introducing any significant complications relative to the standard procedure outlined in the previous section.

The idea is to replace the truncation condition (2.19) by

$$\lambda_M = \lambda \quad (3.1)$$

for some M , where λ is a constant. This termination condition closes the flow equations hierarchy at level M by introducing the constant λ . The hierarchy closes, because only the first M differential equations are non-trivial if (3.1) is imposed. The equations at levels M and above become algebraic. The termination condition (3.1) does not set the higher order flow parameters to zero: they are instead expressed in

terms of the lower order parameters. For example (2.17) and (3.1) imply

$$\lambda_{M+1} = \lambda \left(M\lambda_0 - (M-1)\lambda_1 \right) . \quad (3.2)$$

Similar relations can be written down for higher flow parameters which generically remain non-vanishing.

The flow equations can be integrated as before for any choice of M and some reasonable set of values of λ . The original subset of inflationary model space is the case of $\lambda = 0$, so clearly all the solutions appearing in the old approach are recovered.

There are in fact two ways to proceed. One option is to integrate the set of $M+1$ nontrivial flow equations. The alternative is to directly solve the termination condition itself. In the case of standard truncation the solution of the truncation condition (2.19) is (2.21). This way H is obtained directly, without going through the flow parameters, in fact circumventing the flow equations themselves. In the case of the modified termination condition one can proceed in the same spirit by expressing (3.1) using (2.15) as

$$(2M_P^2)^M \frac{(H')^{M-1}}{H^M} \frac{d^{M+1}H}{d\phi^{M+1}} = \lambda . \quad (3.3)$$

This is a single differential equation of order $M+1$ which replaces Liddle's (2.20). For $M > 1$ this equation is nonlinear, and one cannot solve it analytically. It is however straightforward to solve numerically. For that purpose it is convenient to write it as a system of first order equations as follows. Introducing

$$H_k \equiv \frac{d^{k+1}H}{d\phi^{k+1}} \quad k = 0, \dots, M \quad (3.4)$$

equation (3.3) can be rewritten as a system of first order differential equations:

$$\begin{aligned} \frac{dH_k}{d\phi} &= H_{k+1} \quad k = 0, \dots, M-1 \\ \frac{dH_M}{d\phi} &= \frac{\lambda}{(2M_P^2)^M} \frac{H_0^M}{H_1^{M-1}} . \end{aligned} \quad (3.5)$$

Supplementing (3.5) with suitable initial conditions one can generalize numerical computations of the type pioneered by Kinney [1] to the wider set of solutions described here. To this end one can express initial values for the H_k in terms of often

used initial values for the standard flow parameters. Indicating initial values by an over-bar one has, from the definition of ϵ_H ,

$$\bar{H}_1 = \pm \bar{H}_0 \sqrt{\frac{\bar{\epsilon}_H}{2M_P^2}} . \quad (3.6)$$

The flow equations (or (3.5)) determine H up to an overall scale, which can be taken as \bar{H}_0 . The choice of sign above reflects the possibility of the inflaton rolling to the left or to the right. Similarly for $k \geq 1$ one can write

$$\bar{H}_{k+1} = \left(\frac{1}{2M_P^2}\right)^k \bar{H}_0 \bar{\lambda}_k \sqrt{\frac{\bar{\epsilon}_H}{2M_P^2}} , \quad (3.7)$$

where $\bar{\lambda}_k$ are initial values of the flow parameters, which can be related directly to those used by Kinney [1]: one has $\bar{\lambda}_1 = \frac{1}{2}(\bar{\sigma}_H + 4\bar{\epsilon}_H)$ where $\bar{\sigma}_H$ is the initial value of Kinney's σ_H , and $\bar{\lambda}_k$ are the initial values of Kinney's ${}^k\lambda_H$ (for $k \geq 1$). One can now numerically integrate the equations (3.5) choosing initial values of $\epsilon_H, \sigma_H, {}^k\lambda_H$ from the same ranges as those used in [1] (and most of the literature devoted to this subject) to facilitate comparison.

The numerical computations following from this prescription will not be presented here; instead the following section will describe some analytic considerations which give a glimpse of space of potentials defined by the procedure introduced above.

4. Some Analytic Results

To understand the difference in the space of potentials scanned by the truncation of the flow equations described in the last section it is instructive to look at the simplest cases, that is when (3.1) is imposed with $M = 0$ and $M = 1$, which are very simple to solve analytically.

In the case $M = 0$ the termination condition (3.1) is the statement that λ_0 is constant. Since λ_0 is just ϵ_H , this is power law inflation (when $\lambda < 1$). The termination condition (3.1) becomes

$$2M_P^2 \left(\frac{H'}{H}\right)^2 = \lambda . \quad (4.1)$$

For this to make sense one can only allow non-negative values of λ . As discussed in the last section, (4.1) can be regarded (in the spirit of [31]) as a first order differential

equation for $H(\phi)$. The general solution is

$$H(\phi) = A \exp(\pm \sqrt{\frac{\lambda}{2M_P^2}} \phi) . \quad (4.2)$$

This involves one integration constant, A . The corresponding potential, obtained from (2.18) is

$$V(\phi) = A^2 M_P^2 (3 - \lambda) \exp(\pm \sqrt{\frac{2\lambda}{M_P^2}} \phi) . \quad (4.3)$$

This is the well known example of Lucchin and Matarrese [32].

This simplest case already shows the difference between the procedure proposed in the previous section and the one normally used in the literature. While the standard procedure is equivalent to scanning over the set of polynomial potentials of some order, here one obtains a non-polynomial one. Clearly, all the Hubble flow parameters are non-zero: they are all given by powers of the constant λ . The case of standard level 0 truncation is obtained in the limit $\lambda \rightarrow 0$, which describes de Sitter expansion.

Imposing the termination condition (3.1) with $M = 1$ is also solvable, and rather interesting. The termination condition reads

$$2M_P^2 \frac{H''}{H} = \lambda . \quad (4.4)$$

Here there is no restriction on the sign of λ , and the character of the solutions of this differential equation depend on this sign. Since the differential equation is of second order there will be two integration constants. The general solution is³

$$H(\phi) = \begin{cases} A \cosh(\sqrt{\frac{\lambda}{2M_P^2}} \phi) + B \sinh(\sqrt{\frac{\lambda}{2M_P^2}} \phi) & \text{for } \lambda > 0 \\ A + B\phi & \text{for } \lambda = 0 \\ A \cos(\sqrt{\frac{|\lambda|}{2M_P^2}} \phi) + B \sin(\sqrt{\frac{|\lambda|}{2M_P^2}} \phi) & \text{for } \lambda < 0 . \end{cases} \quad (4.5)$$

The case $\lambda = 0$ is of course the result of the standard truncation at this level.

For a sensible inflationary solution one has to make a choice of integration constants and restrict the range of ϕ appropriately so as to ensure that H' does not

³Potentials of this type were previously considered in references [23, 24]. They can all be characterized as models with constant η_H .

change sign. Rather than discuss this further, this presentation will focus on one special case, that of $\lambda > 0$ with $B = 0$:

$$H(\phi) = A \cosh\left(\sqrt{\frac{\lambda}{2M_P^2}}\phi\right). \quad (4.6)$$

The Hubble slow-roll parameters are

$$\epsilon_H = \lambda \tanh^2\left(\sqrt{\frac{\lambda}{2M_P^2}}\phi\right) \quad (4.7)$$

$$\eta_H = \lambda. \quad (4.8)$$

The corresponding potential, which follows from (2.18) reads

$$V(\phi) = A^2 M_P^2 \left(\lambda + (3 - \lambda) \cosh^2\left(\sqrt{\frac{\lambda}{2M_P^2}}\phi\right) \right). \quad (4.9)$$

Note that for $\lambda = 3$ the potential becomes constant: this solution was discussed by Tsamis and Woodard [26] under the name ultra-slow-roll inflation. It was later analysed by Kinney as an example where the spectrum of curvature perturbations is exactly scale invariant but where the horizon-crossing formalism fails [25]. The solution (4.6) is valid for a range of λ , so taking λ close to 3, but not exactly 3, should provide interesting examples with almost scale invariant spectra⁴. Since the potential (4.9) in this case is not polynomial, such examples are outside the realm of standard truncated flow equation simulations.

Terminating the hierarchy using (3.1) with $M > 1$ involves solving a nonlinear equation of order $M + 1$. The general solution, depending on $M + 1$ constants of integration appears not to be available analytically, but one special solution is easy to write down: it is

$$H(\phi) = A \exp\left(\pm \frac{1}{\sqrt{2M_P^2}} \lambda^{\frac{1}{2M}} \phi\right). \quad (4.10)$$

Both choices of sign are admissible, but one cannot take linear combinations, as the equation is not linear. Therefore for higher-level solutions one needs to resort to numerical integration, as discussed in the previous section.

⁴A very interesting and nontrivial class of potentials with exactly scale invariant spectra was obtained recently by Starobinsky [33]. It appears that these models are not of the type considered here.

5. Conclusions

The inflationary flow equations are the basis of a very widely used approach to exploring the realm of inflationary scalar field theories. The standard method of truncating the infinite hierarchy of flow equations restricts the class of scalar potentials to polynomials in the inflaton field. This paper presented a different, more general, way of solving the hierarchy. The resulting class of potentials includes those obtained by the standard truncation method, but is much broader in that it also includes a wide range of non-polynomial potentials. The procedure boils down to solving the system of differential equations (3.5), which is the main result presented here.

Some insight into the space of potentials accessible using this method can be gleaned from analytic solutions to lowest level termination conditions. As discussed in section 4, examples found this way have already appeared in the literature in various contexts. Here they serve the purpose of illustrating how the new solutions of the flow equations extend the range of potentials scanned.

The approach described here could be particularly useful for exploring models of inflation involving potentials with “features”, where the potential is non-polynomial in an essential way [19]-[22]. It would also be interesting to understand the impact of the present work on numerical flow equation simulations used to analyze data from WMAP and other sources [4, 10, 11].

As this paper was being written up the generalization of the flow equations to the case of DBI models was introduced by Peiris et al. [13]. The new procedure presented here can be used without change also in that case. Due to the occurrence of the “Lorentz” factor γ in the Dirac-Born-Infeld action, the hierarchy of flow equations requires two truncation conditions, which Peiris et al. solved, demonstrating that their method scans the space of $V(\phi)$ and $\gamma(\phi)$ which are polynomials in the inflaton field. Replacing the two truncation conditions used in [13] by conditions of the sort advocated here works in the same way as the canonical case discussed in this note. It would clearly be of interest to investigate how modifying that study along these lines affects the results reported there, given the tight observational constraints on that class of models.

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